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ON LOCAL ERGODIC CONVERGENCE OF SEMI-GROUPS AND ADDITIVE PROCESSES

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ABSTRACT

We prove the local ergodic theorem in L_∞: Let $\{T_t\}_{t>0}$ be a strongly continuous semi-group of positive operators on L_1. If T_t is continuous at 0, then $\varepsilon^{-1}\int_0^\varepsilon T_t^* f(x)dt \to T_0^* f(x)$ a.e., for every $f \in L_\infty$. The technique shows how to obtain the L_p local ergodic theorems from the L_1-contraction case. It applies also to differentiation of L_p additive processes. The n-dimensional case, which is new, is proved by reduction to the n-dimensional L_1-contraction case, solved by M. Akcoglu and A. del Junco.

1. Introduction

N. Wiener [22] proved a local ergodic theorem for measure preserving flows in 1939. Thirty years later, many authors started to be interested in obtaining local ergodic theorems for semi-groups. Thus the problem is the following: Let (X, Σ, m) be a probability space, and let $\{T_t\}_{t>0}$ be a strongly continuous (at $t > 0$) semi-group of bounded linear operators in L_p $(1 \leqq p < \infty)$. When do we have that $\varepsilon^{-1}\int_0^\varepsilon T_t f(x)dt$ converges a.e., as $\varepsilon \to 0^+$, for every $f \in L_p$? (To be more precise, is there some f_0 such that $\varepsilon_n^{-1}\int_0^{\varepsilon_n} T_t f(x)dt \to f_0(x)$ a.e., as $\varepsilon_n \to 0^+$, or, equivalently [3], [20], are there representatives of $\varepsilon^{-1}\int_0^\varepsilon T_t f(x)dt$ such that the limit exists a.e. as $\varepsilon \to 0^+$.)

For L_∞ we have a similar problem, but $\{T_t\}$ is assumed only w*-continuous at $t > 0$, and each T_t is w*-continuous on L_∞ (i.e., $\{T_t\}$ is the dual of an L_1-continuous semi-group). A generalization of Wiener's result in L_∞, for non-singular transformations, was given by U. Krengel [10].

In this paper we are interested in the case of a semi-group of *positive* operators (i.e., $f \geqq 0 \Rightarrow T_t f \geqq 0$), and we assume "*local boundedness*": $\sup\{\| T_t \| : 0 < t \leqq 1\} < \infty$.

The local ergodic theorem was proved in the following cases, for $\{T_t\}$ positive:

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(A) $p = 1$, $\|T_t\| \leqq 1$, $T_t \xrightarrow[t\to 0^+]{} I$ *strongly* (Krengel [9], Ornstein [15]).

(B) $p = 1$, $\|T_t\| \leqq 1$ (Akcoglu–Chacon [1]).

(C) $1 \leqq p < \infty$, $T_t \xrightarrow[t\to 0^+]{} I$ *strongly* (Kubokawa [11], [12], McGrath [23]).

(D) $1 \leqq p < \infty$, $T_t \xrightarrow[t\to 0^+]{} E$ *strongly*, $\|E\| \leqq 1$ (Sato [16]).

In section 2 we solve the L_∞ case, and show how to obtain the result (D) of Sato by reduction to (A) or (B).

Sato [18], and Akcoglu and Krengel [5], have shown that for $p = 1$ continuity at 0 is not sufficient for the local ergodic theorem to hold (nor is it necessary [1]). For $1 < p < \infty$, Sato [16] has shown that local boundedness implies the continuity at 0, but the question of the local ergodic theorem in L_p is still unresolved.

Akcoglu and Krengel [3] have generalized the result of [1] to obtain a differentiation theorem for additive processes in L_1, with respect to a positive contraction semi-group. In [4] they make a refinement to obtain a result in L_p. Our method shows how to obtain their L_p result from the L_1 result. This is done in detail for n-dimensional processes in section 3, where the L_1 contraction case was proved by M. Akcoglu and A. del Junco [2]. A local ergodic theorem for n-dimensional semi-groups is in [21].

Finally, we mention that the local ergodic theorem was proved for a contraction semi-group (not necessarily positive) in L_1, under the assumption that $T_t \to I$ strongly as $t \to 0$, by Kubokawa [13], Kipnis [8], and Sato [19]. A partial result in L_∞ was given by Sato [20]. It is not clear how to apply our result (for the positive case) to the general case. For contractions in L_p which are also L_∞ contractions see [2], [11], while a negative answer for general contraction semi-groups in L_2 is given in [5] and in [24].

2. The local ergodic theorem in L_∞

It is now known that a semi-group of positive linear operators on L_1 may be continuous at 0 and fail the local ergodic theorem (Sato [18], Akcoglu and Krengel [5]). On the other hand, the result of Akcoglu and Chacon [1], and an example there, show that continuity at zero is not necessary. We show that in L_∞ the situation is different.

THEOREM 2.1. *Let $\{T_t\}_{t>0}$ be a strongly continuous (at $t > 0$) semi-group of positive linear operators on $L_1(X, \Sigma, m)$. Then the following are equivalent:*

(a) $\sup_{0<t\leqq 1} \|T_t\| < \infty$, *and* $\varepsilon^{-1}\int_0^\varepsilon T_t^* f(x)dt$ *converges a.e. for every* $f \in L_\infty$.

(b) *$\{T_t\}$ is strongly continuous at zero (i.e., there is a T_0 such that $T_t \xrightarrow[t\to 0]{} T_0$ strongly in L_1.*

When (a) *and* (b) *are satisfied, the limit in* (a) *is necessarily* $T_0^* f(x)$.

PROOF. (a) $\Rightarrow$ (b). Let $u \in L_1$. Then, by Lebesgue's theorem and (a), $\langle \varepsilon^{-1}\int_0^\varepsilon T_t u, f\rangle = \langle u, \varepsilon^{-1}\int_0^\varepsilon T_t^* f dt\rangle$ converges for every $f \in L_\infty$. By the weak sequential completeness of L_1, there is an element $T_0(u)$ such that weak-$\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_0^\varepsilon T_t u dt = T_0(u)$. The mapping is clearly linear and positive.

We can now apply theorem 10.5.1. of [7] to obtain that T_0 is a projection on $L_0 = \overline{\bigcup\{T_t L_1 : t > 0\}}$, and $T_0 T_t = T_t T_0 = T_t$ for $t > 0$ (only weak convergence is actually needed, once we know that the weak limit is in L_1, since each T_t is also continuous on L_1 with its weak topology). For $t > 0$, on $T_t L_1$ we have $\lim_{s\to 0} T_s T_t u = T_t u$ by strong continuity, and since $\sup_{0<s\leq 1}\| T_s \| < \infty$, $\lim_{t\to 0} T_t u = u$ on L_0, hence $T_t v = T_t T_0 v \xrightarrow[t\to 0]{} T_0 v$, for every $v \in L_1$.

(b) $\Rightarrow$ (a). Take $u \in L_1$ with $u > 0$ a.e., and define $C = \{x : T_0 u(x) > 0\}$. Then for $0 \leqq v \in L_1$ we have $\{T_0 v > 0\} \subset C$ (so the definition of C does not depend on the choice of u). For $t > 0$ and $v \in L_1$ we have $T_t v = T_0 T_t v \in L_1(C)$, so $L_1(C)$ is invariant under $\{T_t\}$. Define $D = X - C$. Then $T_t^* 1_D \equiv 0$ for $t \geqq 0$, since for $v \in L_1$ we have

$$\langle v, T_t^* 1_D\rangle = \langle T_t v, 1_D\rangle = 0.$$

Denote $S_\varepsilon = \varepsilon^{-1}\int_0^\varepsilon T_t dt$ (which exists in the strong L_1-topology). We have to show that $S_\varepsilon^* f \xrightarrow[\varepsilon\to 0]{} T_0^* f$ a.e., and we show first that the convergence holds on C. Since $T_t^* 1_D = 0$ for every $t \geqq 0$, we need consider $f \in L_\infty(C)$.

Let R_t be the restriction of T_t to $L_1(C)$. Then for $v \in L_1(C)$, $f \in L_\infty(C)$, we have $\langle v, R_t^* f\rangle = \langle R_t v, f\rangle = \langle T_t v, f\rangle = \langle v, T_t^* f\rangle$, so that $R_t^* f = T_t^* f$ on C. Let $\alpha > 0$ be greater than the type of $\{T_t\}$.

Let $u > 0$ a.e. be in L_1, and let $u_0 = \int_0^\infty e^{-\alpha t} T_t u dt$. By continuity $u_0 > 0$ on C, and $u_0 \in L_1(C)$ since $T_t u \in L_1(C)$.

Now, for $t \geqq 0$ we have

$$R_t u_0 = T_t u_0 = \int_0^\infty e^{-\alpha t} T_{t+s} u ds \leqq e^{\alpha t} u_0.$$

Hence, for $t \geqq 0$ we have, for $f \in L_\infty(C)$, that

$$\int (e^{-\alpha t} R_t^* f) u_0 dm = \int e^{-\alpha t} f R_t u_0 dm \leqq \int f u_0 dm,$$

so that $e^{-\alpha t} R_t^*$ is a contraction semi-group on $L_1(C, u_0 dm)$. Let $g \in L_\infty(C)$, $f \in L_\infty(C)$. Then

$$\int g(e^{-\alpha t}R_t^* f)u_0 dm = \int e^{-\alpha t}R_t(gu_0)f dm = e^{-\alpha t}\int T_t(gu_0)f dm$$

$$\xrightarrow[t\to 0]{} \int T_0(gu_0)f dm = \int g(R_0^* f)u_0 dm.$$

By approximation $e^{-\alpha t}R_t^*$ is weakly continuous at 0, hence strongly continuous at 0 on $L_1(C, u_0 dm)$. We can now apply the local ergodic theorem [9] to obtain that for $f \in L_\infty(C) \subset L_1(C, u_0 dm)$, $\varepsilon^{-1}\int_0^\varepsilon e^{-\alpha t}R_t^* f(x)dt$ converges to $R_0^* f(x)$ a.e. on C. Hence also $S_\varepsilon^* f(x) \xrightarrow[\varepsilon\to 0]{} T_0^* f(x)$ a.e. on C (since $e^{-\alpha t} \to 1$).

We now have to prove convergence on D, for $f \in L_\infty(C)$. Let $f_1 = q\text{-}\limsup_{\varepsilon\to 0} S_\varepsilon^* f$, $f_2 = q\text{-}\liminf_{\varepsilon\to 0^+} S_\varepsilon^* f$ (where q-limit means that $\varepsilon \to 0^+$ along a countable set).

We know that $\|T_t\| \leqq Me^{\alpha t}$ by strong continuity, and hence $\|S_\varepsilon^* f\|_\infty \leqq \|f\|_\infty \sup_{0<t\leq 1}\|T_t\|$ for $\varepsilon \leqq 1$. Hence f_1 and f_2 are in L_∞, and

$$T_0^* f_1 \geqq \text{q-}\limsup_{\varepsilon\to 0^+} \varepsilon^{-1}\int_0^\varepsilon T_0^* T_t^* f(x)dt = f_1,$$

and $T_0^* f_2 \leqq f_2$. Hence $f_1 - f_2 \geqq 0$ is supported on D (since $f_1 = f_2$ on C by the previous arguments), and $T_0^*(f_1 - f_2) \geqq f_1 - f_2 \geqq 0$. But $T_0^* 1_D = 0$, hence $f_1 = f_2$. This shows that $S_\varepsilon^* f(x)$ converges a.e. (see [3]). By part (a) the limit is necessarily $T_0^* f(x)$.

Remark. The difficulty in using the above proof also for the L_p case lies in passing from C to D (where continuity at 0 was used), since we need $\sup_{0<\varepsilon\leq 1}\varepsilon^{-1}\int_0^\varepsilon T_t^* f(x)dt \in L_p$ (when T_t^* acts on L_p). A dominated ergodic estimate for power-bounded positive operators in L_p $(1 < p)$ will yield the required result.

We next indicate how to obtain a general form of the local ergodic theorem by reduction to the Akcoglu–Chacon theorem. We assume $1 \leqq p < \infty$, and $\{T_t\}$ a locally bounded semi-group of positive linear operators on L_p. Let $u \in L_p$ satisfy $u > 0$ a.e.

Since $\{T_t\}$ is locally bounded, $\|T_t\| \leqq Me^{\beta t}$ for some $\beta \geqq 0$. Let $\alpha > \beta$, $u_0 = \int_0^\infty e^{-\alpha t}T_t u dt$, and *define* $C = \{u_0 > 0\}$, $D = X - C$. Similarly, let $g \in L_p^*$ satisfy $g > 0$ a.e., $g_0 = \int_0^\infty e^{-\alpha t}T_t^* g dt$ (defined in the weak-* topology of L_∞ if $p = 1$) and define $C^* = \{g_0 > 0\}$, $D^* = X - C^*$.

Theorem 2.2. *For every $f \in L_p$, $\lim_{\varepsilon\to 0^+}\varepsilon^{-1}\int_0^\varepsilon T_t f(x)dt$ exists a.e. on $C^* \cup D$.*

Sketch of Proof. For $f \in L_p$ with $|f| \leqq Ku$, we have that $T_t f = 0$ a.e. on D, for each $t > 0$, since $\langle T_t u, 1_D\rangle = 0$ on $(0, \infty)$. On C^* we use the reduction to the L_1-contraction case, as in the proof of Theorem 2.1 (replacing C there by C^*).

COROLLARY 2.3. *If* $1 \leqq p < \infty$, *and* $\{T_t\}_{t \geqq 0}$ *is a strongly continuous semi-group of positive operators with* $\lim_{t \to 0^+} T_t = E$, $\|E\| \leqq 1$, *then the local ergodic theorem holds.*

PROOF. (a) $1 < p < \infty$. E is a positive contractive projection of L_p. Now $Eu_0 = u_0$ for $u_0 \in L_p$ implies $E^*(u_0^{p-1}) = u_0^{p-1}$, and $C = C^*$.

(b) $p = 1$. Now E is a positive contraction of L_1, having a fixed point supported on C. Hence C is (in) the conservative part of E, and $E^*1_C \geqq 1_C$ yields $C \subset \{E^*1 > 0\} = C^*$. (In fact, after a change of measure E is a conditional expectation [14], but this fact is not needed.)

REMARKS. (1) For $p = 1$, we need not have $C = C^*$. On $L_1(\{1,2\})$ with counting measure define $T_t(f_1, f_2) = (f_1 + f_2, 0)$. Then $C = \{1\}$, $D = \{2\}$. But $T_t^*(g_1, g_2) = (g_1, g_1)$, so $C^* = \{1,2\}$ and $D^* = \varnothing$. (The condition $p = 1$ was inadvertently dropped in the remarks of [4, p. 33].)

(2) The above example shows that the remark $D \subset D^*$ made in [4, p. 33] is incorrect.

3. *n*-dimensional semi-groups and additive processes

We start by generalizing the decomposition given in the previous section. In this section $\{T_t\}_{t \in \mathbf{P}_n}$ is an n-parameter semi-group of positive linear operators on L_p, where $t = (t_1, \cdots, t_n)$ satisfies $t_i > 0$. We assume continuity at t, which means that the n semi-groups $\{T_{t_i e_i}\}_{t_i > 0}$ are continuous (e_i is the i-th unit vector in $\mathbf{R}_n$), and local boundedness, which yields the existence of a β and an $M > 0$ such that $\|T_t\| \leqq Me^{\beta\varphi(t)}$, where $\varphi(t) = \sum_{i=1}^n t_i$. We may and do assume $\beta \geqq 0$.

PROPOSITION 3.1. *Let* $\{T_t\}_{t \in \mathbf{P}_n}$ *be as above. Let* $f \in L_p$ *satisfy* $f > 0$ *a.e. and for* $\alpha > \beta$ *define* $f_0 = \int_0^\infty \cdots \int_0^\infty e^{-\alpha\varphi(t)} T_t f dt$. *Let* $C = \{f_0 > 0\}$ *and* $D = X - C$. *Then:*

(i) $T_t(L_p(X)) \subset L_p(C)$ *for every* $t \in \mathbf{P}_n$,

(ii) $e^{-\alpha\varphi(t)} T_t f_0 \leqq f_0 \in L_p$ *for every* $t \in \mathbf{P}_n$,

(iii) *if* $g \in L_q(D)$, *then* $T_t^* g \equiv 0$, *for* $t \in \mathbf{P}_n$.

PROOF. (ii) Computation.

(iii) If $0 \leqq g \in L_q(D)$, then

$$0 = \langle f_0, g \rangle = \int_0^\infty \cdots \int_0^\infty e^{-\alpha\varphi(t)} \langle f, T_t^* g \rangle dt.$$

Hence the continuous function $\langle f, T_t^* g \rangle = \langle T_t f, g \rangle$ is zero on $\mathbf{P}_n$, and, since $f > 0$ a.e., $T_t^* g = 0$ for $t \in \mathbf{P}_n$.

(i) Let $h \in L_p$. Then $\langle T_t h, g\rangle = \langle h, T_t^* g\rangle = 0$ for $g \in L_q(D)$, by (iii). Hence $T_t h = 0$ a.e. on D.

REMARK. Property (i) shows that the decomposition does not depend on the choice of $f > 0$ a.e. Property (iii) shows also that it does not depend on $\alpha > \beta$ chosen.

Carrying out the same construction for the dual semi-group $\{T_t^*\}$, we obtain a decomposition into sets C^* and D^* (for $p = 1$, $q = \infty$, and weak-$*$ continuity is sufficient. The semi-group is the dual of an L_1 semi-group. Also (iii) is to be read with $g \in L_1$ when the proposition is applied to $p = \infty$).

Terrell [21] proved the local ergodic theorem in L_1 for $\| T_t \| \leqq 1$ and $\lim_{t \to 0} T_t = I$. For $1 < p < \infty$ see [23]. All the results of section 2 can be generalized with similar proofs. We will carry it out in a more general context.

DEFINITION. Let $\{T_t\}_{t \in \mathbf{P}_n}$ be a semi-group as above. Let $\mathcal{T}_n$ be the collection of all order intervals in $\mathbf{R}_n^+$. A set function $F : \mathcal{T}_n \to L_p$ is an *additive process* (with respect to $\{T_t\}$) if:

(3.1) $$T_t F(I) = F(t + I) \qquad \text{for } t \in \mathbf{P}_n, \quad I \in \mathcal{T}_n.$$

For $I_1, \cdots, I_k \in \mathcal{T}_n$ pairwise disjoint such that $\bigcup_{i=1}^{k} I_i \in \mathcal{T}_n$,

(3.2) $$F(I) = \sum_{i=1}^{k} F(I_i).$$

If there is a K such that $\|F(I)\| \leqq K\lambda(I)$ the process is called *bounded* (λ is Lebesgue's measure on $\mathbf{R}_n$).

We denote by $[a, b]$ the order interval $\{t \in \mathbf{R}_n : a \leqq t \leqq b\}$.

THEOREM 3.2. *Let $\{T_t\}_{t \in \mathbf{P}_n}$ be a locally bounded strongly continuous semi-group of positive linear operators on L_p, and let $F : \mathcal{T}_n \to L_p$ be a bounded additive process. Then $\lim_{\varepsilon \to 0^+} \varepsilon^{-n} F[0, \varepsilon(1, 1, \cdots, 1)]$ exists a.e. on $C^* \cup D$.*

REMARK. Akcoglu and del Junco [2] proved that for $p = 1$ and $\| T_t \| \leqq 1$ convergence holds a.e. This result will be used in the proof.

PROOF. We first note that on D the process is zero, i.e. $1_D F(I) = 0$ a.e. This follows from Proposition 3.1 (i), with a proof as in [2, Lemma 2.2], so we need only prove convergence on C^*.

We apply Proposition 3.1 (i) to $\{T_t^*\}$ and obtain that $L_q(C^*)$ is invariant under $\{T_t^*\}$. Denote $S_t = T_t^* |_{L_q(C^*)}$. Then for $f \in L_p(C^*)$, $g \in L_q(C^*)$, we have

$$\langle S_t^* f, g\rangle = \langle f, T_t^* g\rangle = \langle T_t f, g\rangle.$$

Hence $S_t^* f = T_t f$ on C^*, for $f \in L_p(C^*)$. Let $g_0 \in L_q(C^*)$ be the function obtained by applying Proposition 3.1 (i) to T_t^* (with f replaced by g). Let $R_t = e^{-\alpha\varphi(t)} S_t^*$, and $d\mu = g_0 dm$. Then $0 \leqq f \in L_p(C^*, m)$ is in $L_1(C^*, \mu)$, and, by Proposition 3.1 (ii), we have

$$\int R_t f d\mu = \int f e^{-\alpha\varphi(t)} T_t^* g_0 dm \leqq \int f g_0 dm = \int f d\mu.$$

Hence $\{R_t\}$ is an n-parameter semi-group of positive linear contractions of $L_1(C^*, \mu)$. If $g \in L_\infty(C^*)$, then $\int (R_t f) g d\mu = e^{-\alpha\varphi(t)} \int f T_t^*(g g_0) dm$, which is continuous at $t > 0$ by the continuity of $\{T_t\}$. Hence $\{R_t\}_{t \in P_n}$ is a continuous n-parameter semi-group in $L_1(C^*, \mu)$.

We first assume $F(I) \geqq 0$ for every I (a positive process).

We now construct a bounded additive process G in $L_1(C^*, \mu)$ (with respect to $\{R_t\}$), using the given process F. Now, for ϕ a continuous function from $\mathbf{R}_n$ to $\mathbf{R}$, with bounded support, $\int \phi(s) dF(s)$ is defined as an element of L_p (see [2, (3.3)]. In fact, for a bounded subset $A \subset \mathbf{R}_n^+$, F defines a vector valued measure, and the integral $\int \phi(s) dF(s)$ is defined for ϕ bounded measurable with compact support. Let $F^*(I) = 1_{C^*} F(I)$. Then $\int \phi(s) dF^*(s)$ is $1_{C^*} \int \phi(s) dF(s)$. With these preliminaries, define the process

$$G(I) = \int_I e^{-\alpha\varphi(s)} dF^*(s) = \int 1_I(s) e^{-\alpha\varphi(s)} dF^*(s)$$

(values restricted to points in C^*). Then, since $\alpha > 0$ and $I \subset \mathbf{R}_n^+$, and F is positive

$$\int_{C^*} G(I) d\mu = \int_{C^*} \left(\int_I e^{-\alpha\varphi(s)} dF^*(s) \right) g_0(x) dm$$

$$\leqq \int_{C^*} F(I) g_0 dm \leqq \|F(I)\|_p \|g_0\|_q \leqq K\lambda(I) \|g_0\|_q.$$

Hence $G(I)$ is countably additive (being an integral) and bounded in $L_1(C^*, \mu)$. We show that $G(I)$ satisfies (3.1) with R_t. The next equalities hold a.e. on C^*:

$$R_t G(I) = e^{-\alpha\varphi(t)} S_t^* \left[\int_I e^{-\alpha\varphi(s)} dF^*(s) \right] = 1_{C^*} e^{-\alpha\varphi(t)} T_t \left[\int 1_I(s) e^{-\alpha\varphi(s)} dF(s) \right]$$

$$= 1_{C^*} e^{-\alpha\varphi(t)} \int 1_I(s-t) e^{-\alpha\varphi(s-t)} dF(s) = 1_{C^*} G(I+t) = G(I+t).$$

(We have used the formula $T_t \int \phi(s) dF(s) = \int \phi(s-t) dF(s)$, which follows from $T_t F(I) = F(I+t)$.)

By the Akcoglu–del Junco theorem [2], $\lim_{\varepsilon \to 0^+} \varepsilon^{-n} G[0, \varepsilon(1, 1, \cdots, 1)]$ exists a.e. on C^*. But $G(I) \leqq F^*(I) = 1_C \cdot F(I)$, and

$$G[0, \varepsilon(1, 1, \cdots, 1)] \geqq e^{-\alpha n \varepsilon} F^*[0, \varepsilon(1, 1, \cdots, 1)].$$

Hence

$$\varepsilon^{-n} G[0, \varepsilon(1; 1, \cdots, 1)] \leqq \varepsilon^{-n} F^*[0, \varepsilon(1, 1, \cdots, 1)] \leqq e^{\alpha n \varepsilon} \varepsilon^{-n} G[0, \varepsilon(1, 1, \cdots, 1)]$$

and the limit exists, since $e^{\alpha n \varepsilon} \to 1$, and the theorem is proved for positive F.

For a general bounded additive process F, the proof in [2, (3.6)] shows that it is the difference of two positive bounded additive processes (L_1 or the contractive nature of $\{T_t\}$ is not used there). Hence the theorem is proved.

Corollary 3.3. *If, in addition to the assumptions of Theorem* 3.2., $\{T_t\}$ *is continuous at the origin* $0 = (0, 0, \cdots, 0)$, *and* $E = \lim_{t \to 0} T_t$ *is a contraction, then the convergence holds a.e.* $(1 \leqq p < \infty)$.

Proof. It can be shown that continuity at 0 gives that C is the support of positive E-invariant functions, as is done in the one-dimensional case (in the course of the proof of Theorem 2.1). Now the proof of Corollary 2.3 yields $C \subset C^*$ (with equality for $1 < p < \infty$).

Remark. For the one-dimensional case and F a positive additive process, boundedness of the process is not required in [3], and our proof yields a proof of [4, theorem 1] by reduction to [3] (since we use boundedness, in the *positive* case, only to obtain the boundedness assumption of [2]).

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